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Simple Harmonic motion

Repetitive motion in nature is called harmonic motion. As per the definition, they repeat their initial positions after fixed interval of times.

When the motion is repetitive as well as its velocity varies in such a way that the retardation depends on the position of the body, the motion is called simple harmonic. In this type of motion, the velocity at the ends is zero and is maximum at the middle or mean position.

Therefore, if x be the displacement of the body away from the mean position and r be the retardation then

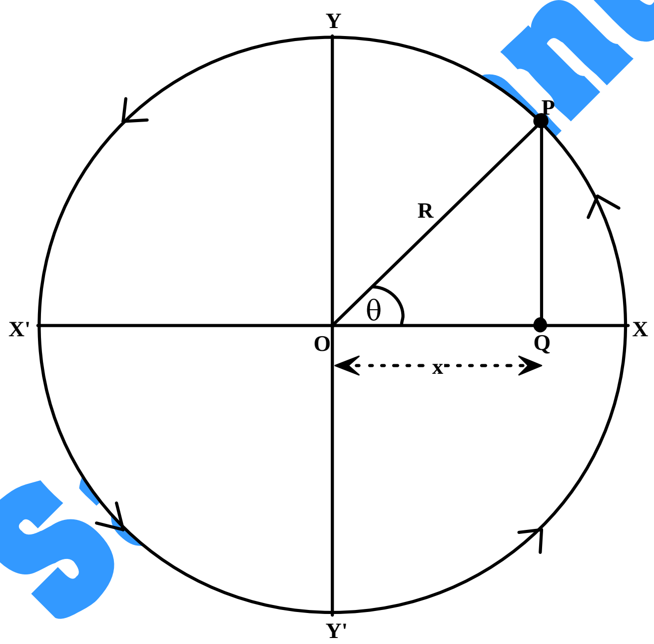
$$r \propto x$$

Since retardation is negative of acceleration i.e. a = - r. Therefore, the expression is written as

$$a \propto -x$$

Motion of the projection of a body moving in a circle

Let a body be uniformly moving in a circle of radius ‘r’ in anticlockwise direction. If two light sources are used above and below the circle will cast the shadows of the body on the x-axis. The distance of this shadow at a certain time from the center is called the ‘projection’ of the body. In this case the shadow is in the x- axis then it is called x-projection.



When the shadow along the x-axis is observed, it is found that the shadow is the fastest at the mean position whereas the velocity decreases towards the sides and becomes zero for a moment to return back.

In the figure, OQ = x, OP = r and $\angle POQ = \theta$

In ΔPOQ, $\cos\theta = \dfrac{x}{r}$

$\therefore x = r\cos\theta$

The measurement of θ at an instant is difficult so, it is expressed in terms of measurable quantities ω & t.

$$\therefore x = r\cos\omega t$$

Differentiating both sides wrt ‘t’ gives

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$$\frac{dx}{dt} = \frac{d(r\cos\omega t)}{dt} \qquad \text{Or,} \quad v = r\frac{d(\cos\omega t)}{dt}$$

Or, $v = r(-\sin\omega t)\omega$

Or, $v = -\omega r\sin\omega t$

Differentiating both sides wrt 't' gives

$$\frac{dv}{dt} = \frac{-d(r\omega\sin\omega t)}{dt} \qquad \text{Or,} \quad a = -\omega r\frac{d(\sin\omega t)}{dt}$$

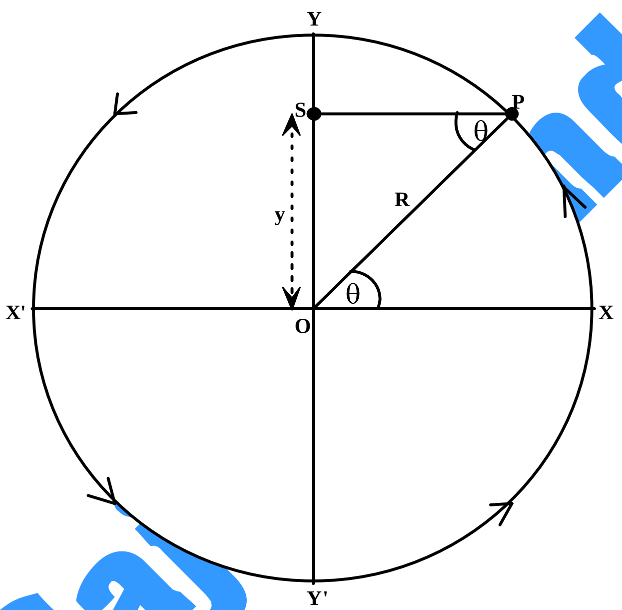
Or, $a = -\omega r\omega\cos\omega t$

Or, $a = -\omega^2 r\cos\omega t$

Or, $a = -\omega^2 x$

Here, the quantity 'ω' is a constant so, **acceleration (a)** $\propto -x$

Therefore, the x-projection of a body moving in a circle with uniform velocity follows simple harmonic motion.



The same derivation can be worked for y-projection also. For this, let, in the figure, OS = y, OP = R, $\angle$OPS = θ

In triangle OSP, $\sin\theta = \frac{y}{R}$

$\therefore y = R\sin\theta$

The measurement of θ at an instant is difficult. So, it is expressed in terms of measurable quantities ω & t.

$\therefore y = R\sin\omega t$

Differentiating both sides wrt t, we get

$$\frac{dy}{dt} = \frac{d(R\sin\omega t)}{dt}$$

Or, $v = R\frac{d(\sin\omega t)}{dt}$

Or, $v = R(\cos\omega t)\omega$ $\qquad\therefore v = \omega R\cos\omega t$

Again differentiating both sides wrt 't' gives

$$\frac{dv}{dt} = \frac{d(\omega R\cos\omega t)}{dt} \qquad \text{Or,} \quad a = \omega R\frac{d(\cos\omega t)}{dt}$$

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$$\textbf{Or,} \quad a = -\omega R\omega \sin\omega t \qquad \textbf{Or,} \quad a = -\omega^2 R \sin\omega t$$

$$\therefore \quad a = -\omega^2 y$$

Here, the quantity ω is a constant. So, $\textbf{acceleration(a)} \propto -\mathbf{y}$.

Time period

It is the total period of time in which an object or the particle covers a certain distance to one complete movement. Time period is the time to cover a complete movement.

The x-projection takes the same time as the revolving body to perform its complete movement. The time period of the revolving body is

$$T = \frac{2\pi}{\omega}$$

$\therefore$ Time period of the x-projection (T_x) = $T = \dfrac{2\pi}{\omega}$

From the relation $\quad \mathbf{a} = -\omega^2 \mathbf{x}, \quad \dfrac{-a}{x} = \omega^2$

$$\textbf{Or,} \quad |\omega^2| = \left|-\frac{a}{x}\right| = \frac{a}{x}$$

$$\omega = \sqrt{\frac{a}{x}}$$

$\therefore$ Time period of the x-projection (T_x) = $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{a/x}} = 2\pi\sqrt{\dfrac{x}{a}}$

$$T_x = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

This relation can be obtained by considering y-projection also.

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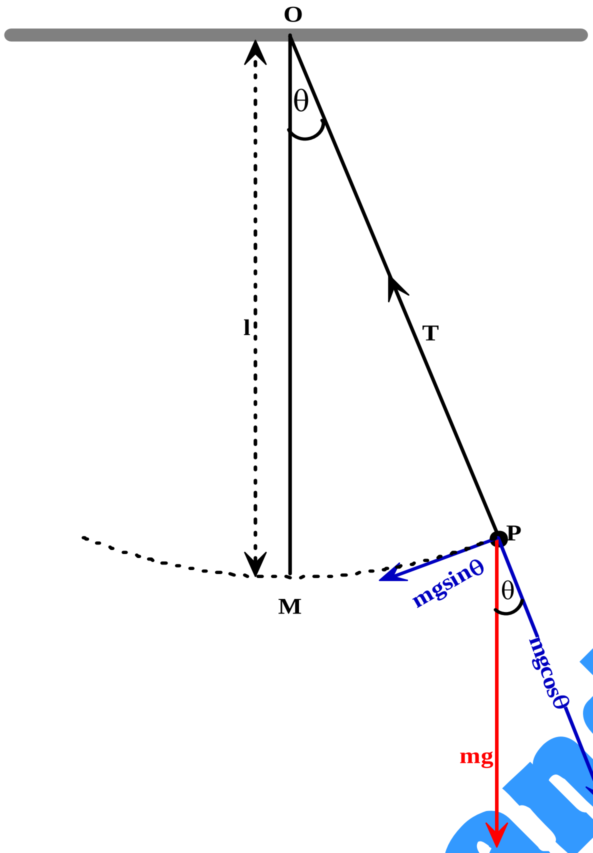
Simple pendulum

A pendulum is a simple body performing repeated oscillations. A simple pendulum is a body of a certain mass attached to an inextensible string and oscillating freely under the influence of gravity from a perfectly fixed point.



The figure shows a certain body of mass 'm' connected to a fixed point by an inextensible string of length 'l' at a certain point of time it is at position P such that the string forms an angle θ with the vertical.

The weight of the body is given by $\mathbf{W = mg}$ which acts vertically downwards. It exerts certain effects in the direction OP and along arc PM also. Its effect along **Arc PM** is equal to $\mathbf{mg\sin\theta}$ and this force is responsible to bring the body to the mean position. So, this is the restoring force.

$$\mathbf{F_r = mg\sin\theta}$$

Since, F = ma according to Newton's second law, $\mathbf{F_r = ma_r}$ (where, a_r is the acceleration towards the mean position = restoring acceleration).

$$\mathbf{ma_r = mg\sin\theta} \qquad \therefore\ \mathbf{a_r = g\sin\theta}$$

Let 'x' be the displacement of the body at position 'P'. A simple pendulum is generally operated in such a way that the value of 'x' is very small. This is done to make it go through a perfect circular arc by preventing any extension of the string.

In such condition, $\theta \approx \mathbf{0}$, which gives $\mathbf{\sin}\theta \approx \theta$.

$$\textbf{So,}\quad \mathbf{a_r = g\theta} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

In the circular arc formed by the position of the body, $\theta = \dfrac{\mathbf{Arc(PM)}}{\mathbf{OM}} = \dfrac{\mathbf{x}}{\mathbf{l}}$

$$\textbf{So,}\quad \mathbf{a_r = g \times \frac{x}{l}}$$

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i.e. $\mathbf{a_r} = \left(\frac{\mathbf{g}}{\mathbf{l}}\right)\mathbf{x}$ ……….. (ii)

For a particular pendulum at a particular location $\frac{\mathbf{g}}{\mathbf{l}}$ is constant so $\mathbf{a_r} \propto \mathbf{x}$.

Therefore, the motion of simple pendulum with small displacement from the mean position follows simple harmonic motion.

Equation 2 gives,

$$\frac{\mathbf{l}}{\mathbf{g}} = \frac{\mathbf{x}}{\mathbf{a_r}} \text{ ……………. (ii)}$$

In any bodies performing SHM, the time period is given by:

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{x}{a_r}}$$

or, $$\boxed{\mathbf{T} = 2\pi\sqrt{\frac{\mathbf{l}}{\mathbf{g}}}} \text{ …………… (iii)}$$

Therefore, the time period of a simple pendulum depends on the length of the pendulum and acceleration due to gravity and it doesn't depend on the mass of the substance.

Conservation of energy in SHM

Let 'v' be the velocity of the body in SHM at a moment. At that time, let the displacement in X-axis be 'x'. Let ω be the angular velocity of the body moving in the circle and 'r' be the radius of the circle.

Then, $$\mathbf{v} = -\omega\sqrt{\mathbf{r}^2 - \mathbf{x}^2}$$

or, $$\mathbf{v}^2 = \omega^2(\mathbf{r}^2 - \mathbf{x}^2)$$

Now, $$\mathbf{KE} = \frac{1}{2}\mathbf{mv}^2 = \frac{1}{2}\mathbf{m}\omega^2(\mathbf{r}^2 - \mathbf{x}^2)$$

or, $$\mathbf{KE} = \frac{1}{2}\mathbf{m}\omega^2\mathbf{r}^2 - \frac{1}{2}\mathbf{m}\omega^2\mathbf{x}^2 \text{ ………. (i)}$$

The potential energy of the particle when it is at 'x' is determined by the amount of work it can do. Therefore,

PE = Work

When it moves a small distance dx, the amount of work it can do is

$$\mathbf{dW} = \mathbf{Fdx} = \mathbf{ma_r dx} = \mathbf{m}\omega^2\mathbf{xdx}$$

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The total work done when the particles travels the whole distance 'x' is given by

$$\mathbf{W} = \int \mathbf{dW} = \int_0^x m\omega^2 x\,dx = m\omega^2 \int_0^x x\,dx = m\omega^2 \frac{x^2}{2}$$

$$\therefore \ \mathbf{W} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 x^2}$$

Therefore, PE of the body at 'x' = $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 x^2}$ ………. (ii)

$$\therefore \text{Total energy (TE)} = \text{PE} + \text{KE}$$

$$= \frac{1}{2}m\omega^2 x^2 - \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 r^2$$

$$\therefore \text{TE} = \frac{1}{2}m\omega^2 r^2$$

Therefore, the total energy of the body performing simple harmonic motion remains constant.

<u>Case I</u>
When the body is at the mean position, x = 0, then

$$\mathbf{KE} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 r^2} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 x^2}$$

$$= \frac{1}{2}m\omega^2 r^2$$

And, $\text{PE} = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 0^2 = 0$

$$\therefore \text{Total energy (TE)} = \text{PE} + \text{KE} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 r^2}$$

<u>Case II</u>
When the body is the extreme position, x = r, then

$$\mathbf{KE} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 r^2} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 x^2} = 0$$

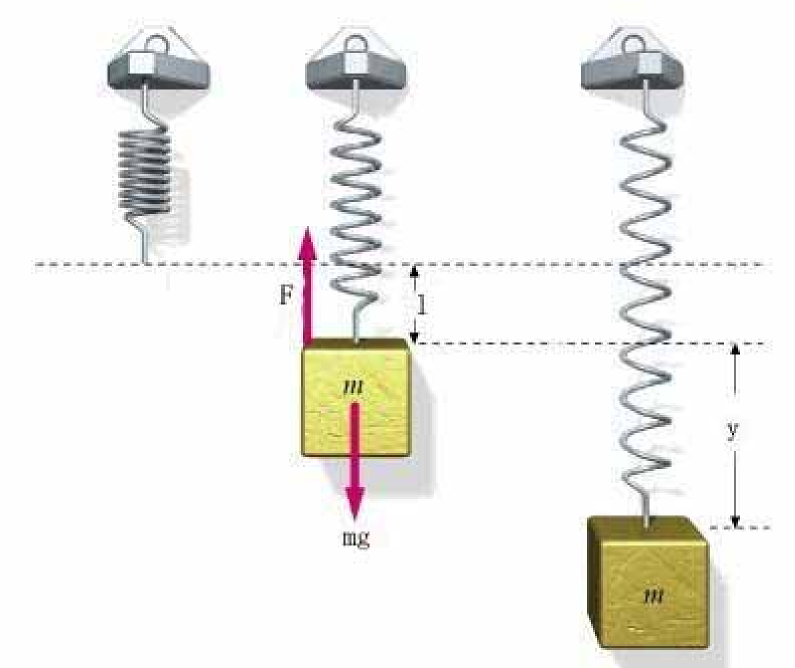
And, $\text{PE} = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 r^2$

$$\therefore \text{Total energy (TE)} = \text{PE} + \text{KE} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m\omega^2 r^2}$$

<u>Motion of a spring-mass system</u>

Let a spring be considered in its normal suspending position which is without any load. To bring it into a working condition it is given certain tightness by suspending a mass m onto it. This will give a downward force:

$$\mathbf{F} = \mathbf{mg} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$



As the spring extends it will develop a restoring force which acts in the upward direction. After some extension the load remains at a constant level, which means the upward and downward forces at that time is equal. The upward force is given by:

$$\mathbf{F} = \mathbf{kl} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Where, k is the spring constant and l is the first extension produced, equation (i) and (ii) gives

$$\text{or,} \quad \mathbf{mg} = \mathbf{kl}$$

$$\therefore \quad \frac{\mathbf{m}}{\mathbf{k}} = \frac{\mathbf{l}}{\mathbf{g}} \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

When the spring is given a further force, it extends further. When it is left free, it will then start to oscillate. This is because of development of additional restoring force. Let F be the final force, then Hooke's law gives

$$\mathbf{F'} = \mathbf{k(1+y)} \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

where, y is the additional extension produced taking the value of total extension to $(1+y)$.

Subtracting (ii) from (iv) gives

$$\mathbf{F'} - \mathbf{F} = k(1+y) - kl$$

$$\text{Or,} \quad \mathbf{F'} - \mathbf{F} = ky + kl - kl$$

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Or, $\mathbf{F' - F = ky}$ .... (v)

Here, the force $\mathbf{F' - F}$ is the additional force which is responsible to create oscillation i.e. produce acceleration in the system.

$\therefore$ $\mathbf{F' - F = ma}$ .... (vi)

Equations (v) & (vi) give

i.e. $\mathbf{ma = ky}$ **Or,** $\mathbf{a = \frac{k}{m} y}$

$\therefore$ $\mathbf{a \propto y}$ .... (vii)

This means acceleration is proportional to the additional displacement produced. So, the motion of such system is simple harmonic.
The relation ma = ky gives

$$\mathbf{\frac{m}{y} = \frac{y}{a}} \quad \text{.................. (viii)}$$

Therefore, time period of the system is given by:

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{y}{a}}$$

Or, $T = 2\pi\sqrt{\frac{m}{k}}$ {from equation (viii)}

Or, $\boxed{\mathbf{T = 2\pi\sqrt{\frac{l}{g}}}}$ {from equation (iii)}

Therefore the time period of such a spring system depends on the extra displacement produced and nothing else.