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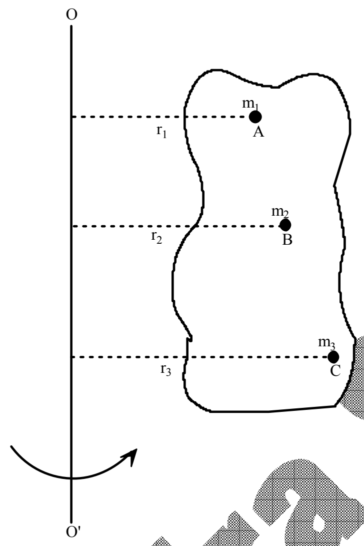
Moment of Inertia

Let there is a multiple particle body which is revolving round a fixed axis with certain angular velocity. Let some particles are considered at positions A, B & C where masses of the particles are m_1, m_2, m_3 and their distances from the axis of rotation are r_1, r_2, r_3 respectively.



Let the kinetic energy of the whole body be calculated. It will be equal to the sum of K.E. of all the small particles.

$$\text{i.e. KE} = KE_A + KE_B + KE_C + \ldots\ldots$$

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots\ldots\ldots$$

(where v_1, v_2, v_3 are the linear velocities of the bodies)

$$= \frac{1}{2}m_1(\omega r_1)^2 + \frac{1}{2}m_2(\omega r_2)^2 + \frac{1}{2}m_3(\omega r_3)^2 + \ldots\ldots\ldots$$

(Here, the angular velocity is same for every particle because they cover equal angle at a certain time whenever they are)

$$\text{so} \quad KE = \frac{1}{2}m_1\omega^2r_1^2 + \frac{1}{2}m_2\omega^2r_2^2 + \frac{1}{2}m_3\omega^2r_3^2 + \ldots\ldots.$$

$$= \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots)\omega^2$$

Here, the term inside the bracket represents the sum of the products of mass & square of distances of all the particles from the axis of rotation. So, the whole term is denoted by,

$$\sum(mr^2) = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots$$

$$KE = \frac{1}{2}\left[\sum(mr^2)\right]\omega^2$$

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If this relationship is compared to the relationship for kinetic energy i.e.

$$KE = \frac{1}{2}mv^2$$

It is found that the term $\sum(mr^2)$ behaves exactly as mass m. It means this quantity should measure the ease or difficulty in rotating a body just like what mass used to measure incase of straight line motion. This quantity is named as moment of inertia or MI and is denoted by 'I'.

$$KE = \frac{1}{2}I\omega^2$$

Going through the relationship of MI, it is found that it depends on the mass and the radius of rotation or revolution of the body.

Relation between Torque and MI of a rigid body

The torque of the rigid body should be equal to the sum of torques of all the small particles i.e.

$$\tau = \tau_1 + \tau_2 + \tau_3 + ...$$

The torque of a certain body is equal to the product of force applied and the perpendicular distance of line of the force from the axis of rotation. For the particles at A, B & C, it is equal to the product of the forces F_1, F_2, F_3 and the radii of rotations r_1, r_2, r_3.

$$\therefore \tau = F_1r_1 + F_2r_2 + F_3r_3 + ...$$

$$Or, \tau = m_1a_1r_1 + m_2a_2r_2 + m_3a_3r_3 + ...$$

(because force is equal to product of masses and acceleration)

$$Or, \tau = m_1(r_1\alpha)r_1 + m_2(r_2\alpha)r_2 + m_3(r_3\alpha)r_3 + ...$$

(Here, α is constant for all particles because they cover same angle whenever they are, thereby making the angular velocity as well as acceleration equal)

$$Or, \tau = m_1r_1^2\alpha + m_2r_2^2\alpha + m_3r_3^2\alpha + ...$$

$$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...)\alpha$$

$$= \left[\sum(mr^2)\right]\alpha$$

$$Or, \tau = I\alpha$$

Therefore, the torque of a rigid body is equal to the product of MI and angular acceleration of the rigid body.

Angular momentum: The angular momentum is the quantity which determines the ease or difficulty in stopping a body which is already rotating or revolving. It is denoted by L.

It depends on the mass of the body, the radius of rotation as well as the linear velocity.

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i.e. $L \propto m$
$L \propto r$
$L \propto v$

Now overall, $L \propto mvr$
Or, $L = kmvr$

In S.I. units, k = 1
So, $L = mvr$

Therefore, angular momentum of a certain body is equal to the product of mass, velocity and the radius of rotation.

Angular momentum of a rigid body: The angular momentum of a multiple particle body is equal to the sum of angular momenta of all the small particles.

$$\begin{aligned} L &= L_1 + L_2 + L_3 + \ldots\ldots\ldots \\ &= m_1v_1r_1 + m_2v_2r_2 + m_3v_3r_3 + \ldots\ldots \\ &= m_1\omega r_1r_1 + m_2\omega r_2r_2 + m_3\omega r_3r_3 + \ldots\ldots \\ &= m_1\omega r_1^2 + m_2\omega r_2^2 + m_3\omega r_3^2 + \ldots\ldots \\ &= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots)\omega \end{aligned}$$

Or, $L = I\omega$

The angular momentum of a rigid body is equal to the product of moment of inertia and angular velocity.

Relation between angular momentum and torque

The relation between angular momentum and moment of inertia is

$$L = I\omega$$

Differentiating both sides wrt 't', we get

$$\frac{dL}{dt} = I\frac{d\omega}{dt} \qquad \text{or,} \quad \frac{dL}{dt} = I\alpha$$

or, $\dfrac{dL}{dt} = \tau$

Therefore, the torque of a rigid body is equal to the rate of change of angular momentum.

However, if a system is isolated (i.e. no external force or torque has been applied from outside), then the torque $\tau = 0$

$$\frac{dL}{dt} = 0 \qquad \text{i.e.} \quad L = \text{constant}$$

$$\therefore I\omega = \text{constant}$$

It means in the absence of the external torque, the angular momentum of a body remains constant i.e. the product of MI and angular velocity of the body is always equal.

If I_1, ω_1 be the MI and angular velocity at the beginning which changes to I_2, ω_2 later, then

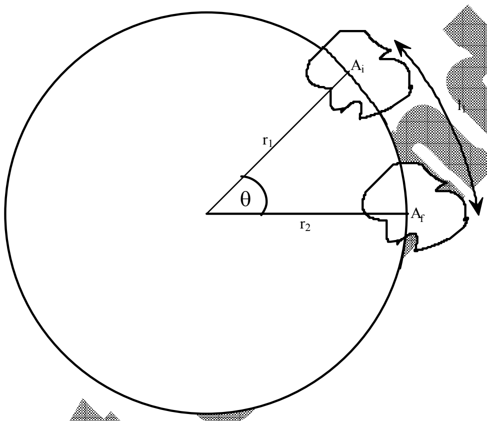
$$\therefore I_1\omega_1 = \text{constant} \qquad \& \qquad I_2\omega_2 = \text{constant}$$

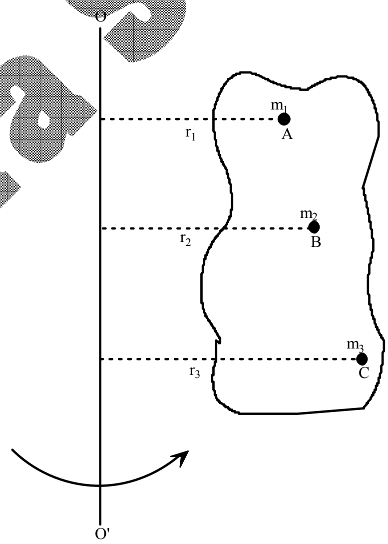
i.e. $\therefore I_1\omega_1 = I_2\omega_2$

This phenomenon is called as Principle of Conservation of Angular Momentum.

Work done by a rigid body

Let the movement of the rigid body shown in the figure be represented from another perspective as shown in the second figure.





In this figure only the motion of body A has been considered which has covered an arc of length l1.

$\therefore$ Work done by this small body at A is given by

$$W_1 = F_1 \times d_1$$
$$= m_1 a_1 l_1 \text{ (the distance traveled by A = arc length it has covered)}$$
$$= m_1(\alpha r_1)(\theta \times r_1)$$
$$= m_1 \alpha r_1^{\,2} \theta$$

Similarly, if individual figures are considered for all the other bodies,

$$W_2 = m_2 \alpha r_2^{\,2} \theta$$

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$$W_3 = m_3 \alpha r_3^2 \theta$$

Therefore, total work done is given by

$$\begin{aligned} W &= W_1 + W_2 + W_3 + \ldots\ldots\ldots \\ &= m_1 \alpha r_1^2 \theta + m_2 \alpha r_2^2 \theta + m_3 \alpha r_3^2 \theta + \ldots\ldots\ldots \\ &= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots)\alpha\theta \\ &= \left[\sum (mr^2)\right]\alpha\theta \\ &= I\alpha\theta \end{aligned}$$

$$\therefore W = \tau\theta \quad (\because I\alpha = \tau)$$

This means the work done by a rigid body is equal to the product of torque and angle covered.

S. No.	Linear quantity or equation	Angular quantity or equation	Relation between linear and angular quantities
1.	Displacement, s	Angular displacement, θ	
2.	Velocity, v	Angular velocity, ω	$v = \omega r$
3.	Acceleration, a	Angular acceleration, α	$a = \alpha r$
4.	Mass, m	Moment of inertia, I	$I = \sum mr^2$
5.	Force, F	Torque, τ	$\tau = F \times r$
6.	$F = ma$	$T = I\alpha$	
7.	$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$	
8.	$P = m \times v$	$L = I \times \omega$	
9.	$F = \frac{dp}{dt}$	$\tau = \frac{dL}{dt}$	
10.	mv = constant ($F = 0$)	$I\omega$ = constant ($\tau = 0$)	
11.	$P = F \times v$	$P = \tau \times \omega$	
12.	$W = F \times s$	$W = \tau \times \theta$	

Rolling of a rigid body along an inclined plane

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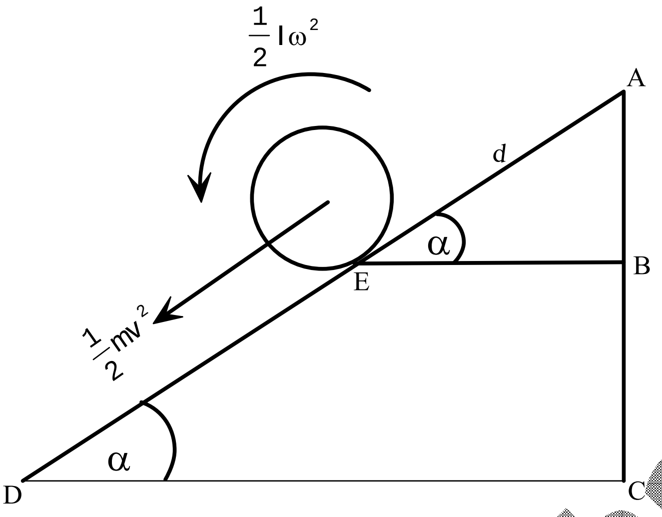

Let a rigid body be allowed to roll along a plane surface inclined at an angle α with the horizontal. When it rolls, it should not be allowed to slip.

Let at any time 't' it reached position E where its linear velocity is 'v' and angular velocity is 'ω'. Here, the normal KE gained by the body is $\frac{1}{2}mv^2$ and rotational KE gained is $\frac{1}{2}I\omega^2$.

$\therefore$ Total K.E. gained is given by

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2}$$

$$= \frac{1}{2}\left(m + \frac{I}{r^2}\right)v^2$$

When the body falls from A to E, the decrease in its height is AB.

$\therefore$ Loss of potential energy $= mg(AB)$

$= mgd\sin\alpha$ (from ΔABE using trigonometry)

According to the law of conservation of energy, the gain of one form of the energy should be equal to loss of another.

$$mgd\sin\alpha = \frac{1}{2}\left(m + \frac{I}{r^2}\right)v^2$$

$$v^2 = \frac{2mgd\sin\alpha}{\left(m + \frac{I}{r^2}\right)}$$

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According to relation, $v^2 = u^2 + 2aS$, the velocity gained by the body at point E is given by:

$$v^2 = 0^2 + 2a_p d \quad \text{(where, a is the acceleration of the body along the plane)}$$

$$= 2a_p d$$

$$\therefore 2a_p d = \frac{2mgd\sin\alpha}{\left(m + \frac{I}{r^2}\right)}$$

Or, $$a_p = \frac{2mgd\sin\alpha}{2d\left(m + \frac{I}{r^2}\right)} \qquad \text{i.e.} \qquad a_p = \frac{mg\sin\alpha}{\left(m + \frac{I}{r^2}\right)}$$

or, $$a_p = \left(\frac{m}{m + I/r^2}\right) g\sin\alpha$$

Here, the term '$g\sin\alpha$' is the acceleration of a small point body descending linearly along the smooth inclined plane surface. Since $\left(\frac{m}{m + I/r^2}\right)$ is definitely less than 'm', it means that the acceleration of a rolling body is less than that of a simply falling body.